

# UNIFORM APPROXIMATION OF POISSON INTEGRALS OF FUNCTIONS FROM THE CLASS $H_\omega$ BY DE LA VALLÉE POUSSIN SUMS

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## Abstract

We obtain asymptotic equalities for least upper bounds of deviations in the uniform metric of de la Vallée Poussin sums on the sets  $C_\beta^q H_\omega$  of Poisson integrals of functions from the class  $H_\omega$  generated by convex upwards moduli of continuity  $\omega(t)$  which satisfy the condition  $\omega(t)/t \rightarrow \infty$  as  $t \rightarrow 0$ . As an implication, a solution of the Kolmogorov–Nikol'skii problem for de la Vallée Poussin sums on the sets of Poisson integrals of functions belonging to Lipschitz classes  $H^\alpha$ ,  $0 < \alpha < 1$ , is obtained.

*MSC 2010:* 42A10

## 1 Introduction

Let  $L$  be the space of  $2\pi$ -periodic summable functions  $f(t)$  with the norm  $\|f\|_L = \int_{-\pi}^{\pi} |f(t)| dt$ , let  $L_\infty$  be the space of  $2\pi$ -periodic functions  $f(t)$  with the norm  $\|f\|_{L_\infty} = \operatorname{ess\,sup}_t |f(t)|$ , and let  $C$  be the space of continuous  $2\pi$ -periodic functions  $f(t)$  in which the norm is defined by the formula  $\|f\|_C = \max_t |f(t)|$ .

Further, let  $C_\beta^q \mathfrak{N}$  be the set of the Poisson integrals of functions  $\varphi$  from  $\mathfrak{N} \subset L$ , i.e., functions  $f$  of the form

$$f(x) = A_0 + \frac{1}{\pi} \int_0^{2\pi} \varphi(x+t) P_{q,\beta}(t) dt, \quad A_0 \in \mathbb{R}, \quad x \in \mathbb{R}, \quad \varphi \in \mathfrak{N}, \quad (1)$$

where

$$P_{q,\beta}(t) = \sum_{k=1}^{\infty} q^k \cos \left( kt + \frac{\beta\pi}{2} \right), \quad q \in (0,1), \quad \beta \in \mathbb{R},$$

is the Poisson kernel with parameters  $q$  and  $\beta$ . If  $\mathfrak{N} = U_\infty$ , where

$$U_\infty = \{\varphi \in L_\infty : \|\varphi\|_{L_\infty} \leq 1\},$$

then the classes  $C_\beta^q \mathfrak{N}$  will be denoted by  $C_{\beta,\infty}^q$ , and if  $\mathfrak{N} = H_\omega$ , where

$$H_\omega = \{\varphi \in C : |\varphi(t') - \varphi(t'')| \leq \omega(|t' - t''|) \quad \forall t', t'' \in \mathbb{R}\},$$

and  $\omega(t)$  is an arbitrary modulus of continuity, then  $C_\beta^q \mathfrak{N}$  will be denoted by  $C_\beta^q H_\omega$ .

Denoting by  $S_k(f; x)$  the  $k$ th partial sum of the Fourier series of the summable function  $f$ , we associate each function  $f \in C_\beta^q H_\omega$  with the trigonometric polynomial of the form

$$V_{n,p}(f; x) = \frac{1}{p} \sum_{k=n-p}^{n-1} S_k(f; x), \quad p, n \in \mathbb{N}, \quad p \leq n. \quad (2)$$

The sums  $V_{n,p}(f; x)$  appeared in [17] and are called the de la Vallée Poussin sums with parameters  $n$  and  $p$ .

The purpose of the present work is to solve the Kolmogorov–Nikol’skii problem for de la Vallée Poussin sums, which consists of obtaining an asymptotic equality as  $n - p \rightarrow \infty$  of the quantity

$$\mathcal{E}(\mathfrak{N}; V_{n,p}) = \sup_{f \in \mathfrak{N}} \|f(\cdot) - V_{n,p}(f; \cdot)\|_C, \quad (3)$$

where  $\mathfrak{N} = C_\beta^q H_\omega$ ,  $q \in (0, 1)$ ,  $\beta \in \mathbb{R}$  and  $\omega(t)$  is a given convex upwards modulus of continuity. Since for  $p = 1$   $V_{n,p}(f; x) = V_{n,1}(f; x) = S_{n-1}(f; x)$ , the quantity

$$\mathcal{E}(\mathfrak{N}; S_{n-1}) = \sup_{f \in \mathfrak{N}} \|f(\cdot) - S_{n-1}(f; \cdot)\|_C$$

is the special case of (3).

The problem of obtaining asymptotic equalities for the quantities of the form

$$\sup_{f \in \mathfrak{N}} \|f(\cdot) - V_{n,p}(f; \cdot)\|_X,$$

has a rich history in various function classes  $\mathfrak{N}$  and metrics  $X \subset L$  and connected with the names of A.N. KOLMOGOROV [2], S.M. NIKOL’SKII [3], A.F. TIMAN [16], S.B. STECHKIN [14], A.V. EFIMOV [1], S.A. TELYAKOVSKII [15], A.I. STEPANETS [10], V.I. RUKASOV [5] and many others. See [4], [7], [8], [9], [11] and [13] for more details on the history of this problem.

S.M. NIKOL’SKII [3] proved that, if  $n \rightarrow \infty$  then

$$\mathcal{E}(C_{\beta,\infty}^q; S_{n-1}) = q^n \left( \frac{8}{\pi^2} \mathbf{K}(q) + O(1)n^{-1} \right), \quad \beta \in \mathbb{R}, \quad (4)$$

where

$$\mathbf{K}(q) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - q^2 \sin^2 t}}, \quad q \in (0, 1)$$

is the complete elliptic integral of the 1st kind. In [14] S.B. STECHKIN improved the estimate of the remainder term in (4) by showing that

$$\mathcal{E}(C_{\beta,\infty}^q; S_{n-1}) = q^n \left( \frac{8}{\pi^2} \mathbf{K}(q) + O(1) \frac{q}{(1-q)n} \right), \quad \beta \in \mathbb{R}, \quad (5)$$

where  $O(1)$  is a quantity uniformly bounded in  $n$ ,  $q$  and  $\beta$ .

In studying approximate properties of the Fourier sums  $S_{n-1}(f; x)$  on the classes  $C_\beta^q H_\omega$ , A.I. STEPANETS [10] proved that for any  $q \in (0, 1)$ ,  $\beta \in \mathbb{R}$  and arbitrary modulus of continuity  $\omega(t)$  the equality

$$\mathcal{E}(C_\beta^q H_\omega; S_{n-1}) = q^n \left( \frac{4}{\pi^2} \mathbf{K}(q) e_n(\omega) + O(1) \frac{\omega(1/n)}{(1-q)^2 n} \right), \quad (6)$$

holds as  $n \rightarrow \infty$ , where

$$e_n(\omega) := \theta_\omega \int_0^{\pi/2} \omega\left(\frac{2t}{n}\right) \sin t \, dt, \quad (7)$$

$\theta_\omega \in [1/2, 1]$ , ( $\theta_\omega = 1$  if  $\omega(t)$  is a convex upwards modulus of continuity) and  $O(1)$  is the same as in (5).

Reasoning as in [10], V.I. RUKASOV and S.O. CHAICHENKO [4] showed that if  $q \in (0, 1)$ ,  $\beta \in \mathbb{R}$ ,  $n, p \in \mathbb{N}$ ,  $p \leq n$  and  $\omega(t)$  is an arbitrary modulus of continuity, then as  $n \rightarrow \infty$ ,

$$\begin{aligned} \mathcal{E}(C_\beta^q H_\omega; V_{n,p}) &= \frac{2q^{n-p+1}}{\pi(1-q^2)p} e_{n-p+1}(\omega) + \\ &+ O(1) \frac{q^{n-p+1}}{p} \omega\left(\frac{1}{n-p+1}\right) \left( \frac{q^p}{1-q^2} + \frac{1}{(1-q)^3(n-p+1)} \right), \end{aligned} \quad (8)$$

where  $O(1)$  is a quantity uniformly bounded in  $n, p, q$  and  $\beta$ .

For the classes  $C_\beta^q H_\omega$  there are well-known estimates of the best uniform approximations by trigonometric polynomials of order not more than  $\leq n-1$  (see, for example, [12, p. 509]):

$$E_n(C_\beta^q H_\omega) = \sup_{f \in C_\beta^q H_\omega} \inf_{t_{n-1}} \|f(\cdot) - t_{n-1}(\cdot)\|_C \asymp q^n \omega(1/n)$$

(the notation  $\alpha(n) \asymp \beta(n)$  as  $n \rightarrow \infty$  means that there exist  $K_1, K_2 > 0$  such that  $K_1 \alpha(n) \leq \beta(n) \leq K_2 \alpha(n)$ ). It's easy to see from (8) that if  $n \rightarrow \infty$  and the values of the parameter  $p$  are bounded, then the de la Vallée Poussin sums realize the order of the best uniform approximation on the classes  $C_\beta^q H_\omega$ . But in that case estimate (8) isn't an asymptotic equality, taking the form

$$\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = O(1) \frac{q^{n-p+1}}{p(1-q^2)} \omega\left(\frac{1}{n-p+1}\right).$$

Thus for  $n-p \rightarrow \infty$  and bounded values of the parameter  $p$  the question of the asymptotic behavior of a principal term of the quantities  $\mathcal{E}(C_\beta^q H_\omega; V_{n,p})$  was still unknown.

The proofs of (6) and (8) are based on the well-known Korneichuk–Stechkin lemma (see, e.g., [10]). In the present work we use a somewhat different way which, as will be seen later, proves itself in some important cases.

## 2 Main result

The main result of the paper is the next theorem.

**Theorem 1.** *Let  $q \in (0, 1)$ ,  $\beta \in \mathbb{R}$ ,  $n, p \in \mathbb{N}$ ,  $p < n$  and let  $\omega(t)$  be a convex upwards modulus of continuity. Then as  $n-p \rightarrow \infty$ :*

$$\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \frac{q^{n-p+1}}{p} \left( \frac{K_{p,q}}{\pi^2} e_{n-p+1}(\omega) + O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)} \right), \quad (9)$$

where

$$K_{p,q} := \int_0^{2\pi} \frac{\sqrt{1 - 2q^p \cos pt + q^{2p}}}{1 - 2q \cos t + q^2} dt, \quad (10)$$

$$e_{n-p+1}(\omega) = \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t dt,$$

$$\delta(p) := \begin{cases} 2, & p = 1, \\ 3, & p = 2, 3, \dots, \end{cases} \quad (11)$$

and  $O(1)$  is a quantity uniformly bounded in  $n, p, q, \omega$  and  $\beta$ .

Since

$$\frac{2}{\pi} \omega\left(\frac{\pi}{k}\right) \leq e_k(\omega) \leq \omega\left(\frac{\pi}{k}\right), \quad k \in \mathbb{N},$$

formula (9) is an asymptotic equality if and only if

$$\lim_{t \rightarrow 0} \frac{\omega(t)}{t} = \infty. \quad (12)$$

An example of moduli of continuity  $\omega(t)$  which satisfy (12), are the functions:

$$\omega(t) = t^\alpha, \quad \alpha \in (0, 1),$$

$$\omega(t) = \ln^\alpha(t+1), \quad \alpha \in (0, 1),$$

$$\omega(t) = \begin{cases} 0, & t = 0, \\ t^\alpha \ln\left(\frac{1}{t}\right), & t \in (0, e^{-1/\alpha}], \quad \alpha \in (0, 1], \\ \frac{1}{\alpha e}, & t \in [e^{-1/\alpha}, \infty), \end{cases}$$

$$\omega(t) = \begin{cases} 0, & t = 0, \\ \ln^{-\alpha}\left(\frac{1}{t}\right), & t \in (0, e^{-(1+\alpha)}], \quad \alpha \in (0, 1], \\ \frac{1}{(1+\alpha)^\alpha}, & t \in [e^{-(1+\alpha)}, \infty). \end{cases}$$

Putting  $\omega(t) = t^\alpha$ ,  $\alpha \in (0, 1)$ , in the hypothesis of Theorem 1 and taking into account that in this case the class  $H_\omega$  becomes the well-known Hölder class  $H^\alpha$ , we obtain the next statement.

**Theorem 2.** *Let  $q \in (0, 1)$ ,  $\beta \in \mathbb{R}$ ,  $n, p \in \mathbb{N}$ ,  $p < n$  and  $\alpha \in (0, 1)$ . Then the following asymptotic equality*

$$\begin{aligned} & \mathcal{E}(C_\beta^q H^\alpha; V_{n,p}) = \\ & = \frac{q^{n-p+1}}{p(n-p+1)^\alpha} \left( \frac{2^\alpha}{\pi^2} K_{p,q} \int_0^{\pi/2} t^\alpha \sin t dt + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)^{1-\alpha}} \right), \end{aligned} \quad (13)$$

is true as  $n-p \rightarrow \infty$ , where  $K_{p,q}$  and  $\delta(p)$  are defined by (10) and (11) respectively, and  $O(1)$  is a quantity uniformly bounded in  $n, p, q, \alpha$  and  $\beta$ .

We note that asymptotic equality (13) for de la Vallée Poussin sums  $V_{n,p}$  with bounded  $p \in \mathbb{N} \setminus \{1\}$  is obtained for the first time.

Asymptotic behavior of the constant  $K_{p,q}$  as  $p \rightarrow \infty$  can be judged from the estimate

$$K_{p,q} = \frac{2\pi}{1-q^2} \left( 1 + O(1)q^p \right), \quad (14)$$

uniform in  $p$  and  $q$  (see [7, p. 130]). The substitution of (14) into equality (9) enables us to obtain the asymptotic estimate of the quantity  $\mathcal{E}(C_\beta^q H_\omega; V_{n,p})$  as  $n-p \rightarrow \infty$  in which the principal term coincides with the first summand in the right-hand side of (8).

In the general case ( $p = 1, 2, \dots, n$ ), as follows from [6, p. 215], the values of the constant  $K_{p,q}$  can be expressed through the values of the complete elliptic integral of the 1st kind  $\mathbf{K}(q^p)$  by means of the following equation:

$$K_{p,q} = 4 \frac{1-q^{2p}}{1-q^2} \mathbf{K}(q^p), \quad p \in \mathbb{N}, \quad q \in (0, 1). \quad (15)$$

Using (15), formulas (9) and (13) have the next representations

$$\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \frac{q^{n-p+1}}{p} \left( \frac{4}{\pi^2} \frac{1-q^{2p}}{1-q^2} \mathbf{K}(q^p) e_{n-p+1}(\omega) + \frac{O(1)\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)} \right), \quad (9')$$

$$\begin{aligned} & \mathcal{E}(C_\beta^q H^\alpha; V_{n,p}) = \\ & = \frac{q^{n-p+1}}{p(n-p+1)^\alpha} \left( \frac{2^{\alpha+2}}{\pi^2} \frac{1-q^{2p}}{1-q^2} \mathbf{K}(q^p) \int_0^{\pi/2} t^\alpha \sin t \, dt + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)^{1-\alpha}} \right). \end{aligned} \quad (13')$$

In addition to Theorem 1 we present the sequent result.

**Theorem 3.** *Let  $q \in (0, 1)$ ,  $\beta \in \mathbb{R}$ ,  $n, p \in \mathbb{N}$ ,  $p < n$  and let  $\omega(t)$  be a convex upwards modulus of continuity. Then as  $n-p \rightarrow \infty$ :*

$$\begin{aligned} & \mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \\ & = \frac{q^{n-p+1}}{p} \left( \frac{4J_{q,p}}{\pi^2} \frac{1-q^p}{1-q} e_{n-p+1}(\omega) + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} \omega\left(\frac{1}{n-p+1}\right) \right), \end{aligned} \quad (16)$$

where the two-sided estimate

$$\frac{1+q^p}{1+q} \mathbf{K}(q^p) \leq J_{q,p} \leq \mathbf{K}(q), \quad (17)$$

holds for  $J_{q,p} = J_{q,p}(n, \omega)$ , and  $e_{n-p+1}(\omega)$ ,  $\delta(p)$  and  $O(1)$  have the same meaning as in Theorem 1.

For  $p = 1$ , as appears from (17),  $J_{q,p} = J_{q,1}(n, \omega) = \mathbf{K}(q)$ . In this case relation (16) becomes the asymptotic equality which is the special case of (6).

### 3 Proof of main result

The proof of Theorem 1 consists of three steps.

*Step 1.* We single out a principal value of the quantity  $\mathcal{E}(C_\beta^q H_\omega; V_{n,p})$ .

To this end we consider first the deviation

$$\rho_{n,p}(f; x) := f(x) - V_{n,p}(f; x), \quad f \in C_\beta^q H_\omega \quad (18)$$

and on the basis of equalities (1) and (2) we write the representation

$$\rho_{n,p}(f; x) = \frac{1}{\pi p} \int_0^{2\pi} \varphi(x+t) \sum_{k=n-p}^{n-1} P_{q,\beta,k+1}(t) dt, \quad (19)$$

in which  $\varphi \in H_\omega$ , and

$$P_{q,\beta,m}(t) = \sum_{j=m}^{\infty} q^j \cos\left(jt + \frac{\beta\pi}{2}\right), \quad m \in \mathbb{N}, \quad q \in (0, 1), \quad \beta \in \mathbb{R}.$$

Since

$$\int_0^{2\pi} \sum_{k=n-p}^{n-1} P_{q,\beta,k+1}(t) dt = 0$$

and, according to [10, p. 118],

$$P_{q,\beta,m}(t) = q^m Z_q(t) \cos\left(mt + \theta_q(t) + \frac{\beta\pi}{2}\right),$$

where

$$Z_q(t) := \frac{1}{\sqrt{1 - 2q \cos t + q^2}},$$

$$\theta_q(t) := \operatorname{arctg} \frac{q \sin t}{1 - q \cos t},$$

it follows from (19) that

$$\rho_{n,p}(f; x) = \frac{1}{\pi p} \int_0^{2\pi} (\varphi(x+t) - \varphi(x)) Z_q(t) \sum_{k=n-p+1}^n q^k \cos\left(kt + \theta_q(t) + \frac{\beta\pi}{2}\right) dt. \quad (20)$$

By virtue of formula (17) in [7, p. 126] the equality

$$\begin{aligned} & \sum_{k=n-p+1}^n q^k \cos\left(kt + \theta_q(t) + \frac{\beta\pi}{2}\right) = \\ & = Z_q(t) q^{n-p+1} \left( \cos\left((n-p+1)t + \frac{\beta\pi}{2}\right) G_{p,q}(t) - \right. \\ & \quad \left. - \sin\left((n-p+1)t + \frac{\beta\pi}{2}\right) H_{p,q}(t) \right) \end{aligned} \quad (21)$$

holds, where

$$\begin{aligned} G_{p,q}(t) &= \cos 2\theta_q(t) - q^p \cos(pt + 2\theta_q(t)), \\ H_{p,q}(t) &= \sin 2\theta_q(t) - q^p \sin(pt + 2\theta_q(t)). \end{aligned}$$

Representing the functions  $G_{p,q}(t)$  and  $H_{p,q}(t)$  in the form

$$G_{p,q}(t) = \frac{\cos(2\theta_q(t) - \theta_{q^p}(pt))}{Z_{q^p}(pt)}, \quad H_{p,q}(t) = \frac{\sin(2\theta_q(t) - \theta_{q^p}(pt))}{Z_{q^p}(pt)},$$

where

$$Z_{q^p}(t) := \frac{1}{\sqrt{1 - 2q^p \cos t + q^{2p}}}, \quad \theta_{q^p}(t) := \arctg \frac{q^p \sin t}{1 - q^p \cos t},$$

from (21) we get

$$\begin{aligned} & \sum_{k=n-p+1}^n q^k \cos \left( kt + \theta_q(t) + \frac{\beta\pi}{2} \right) = \\ & = q^{n-p+1} \frac{Z_q(t)}{Z_{q^p}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{q^p}(pt) + \frac{\beta\pi}{2} \right). \end{aligned} \quad (22)$$

Relations (20) and (22) imply the following equality

$$\begin{aligned} & \rho_{n,p}(f; x) = \\ & = \frac{q^{n-p+1}}{\pi p} \int_0^{2\pi} (\varphi(x+t) - \varphi(x)) \frac{Z_q^2(t)}{Z_{q^p}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{q^p}(pt) + \frac{\beta\pi}{2} \right) dt. \end{aligned} \quad (23)$$

The right-hand side of (23) is 4-periodic in  $\beta$ . Therefore, it will be assumed below that  $\beta \in [0, 4)$ .

Since for any  $\varphi \in H_\omega$  the function  $\varphi_1(u) = \varphi(u+h)$ ,  $h \in \mathbb{R}$ , also belongs to  $H_\omega$ , then from (23) we have

$$\begin{aligned} & \mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \\ & = \frac{q^{n-p+1}}{\pi p} \sup_{\varphi \in H_\omega} \left| \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^2(t)}{Z_{q^p}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{q^p}(pt) + \frac{\beta\pi}{2} \right) dt \right|, \end{aligned} \quad (24)$$

where

$$\Delta(\varphi, t) := \varphi(t) - \varphi(0).$$

Denote by  $\mathcal{J}_{n,p,q,\beta}(\varphi)$  the integral in the right-hand side of (24), that is

$$\mathcal{J}_{n,p,q,\beta}(\varphi) = \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^2(t)}{Z_{q^p}(pt)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{q^p}(pt) + \frac{\beta\pi}{2} \right) dt. \quad (25)$$

Our further goal is to find an asymptotic estimation of the integral  $\mathcal{J}_{n,p,q,\beta}(\varphi)$  as  $n-p \rightarrow \infty$ . For this reason, without loss of generality, we shall assume that the numbers  $n$  and  $p$  have been chosen such that

$$n-p \geq \frac{6}{1-q}. \quad (26)$$

First we show that

$$\begin{aligned} \mathcal{J}_{n,p,q,\beta}(\varphi) &= \\ &= \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^4(t)}{Z_{q^p}(pt) Z_{q,n,p}^2(t)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{q^p}(pt) + \frac{\beta\pi}{2} \right) dt + \\ &\quad + O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)}, \end{aligned} \quad (27)$$

where

$$Z_{q,n,p}(t) := \frac{Z_q(t)}{\sqrt{\frac{n-p+1+2q(\cos t - q)Z_q^2(t) - pq^p(\cos pt - q^p)Z_{q^p}^2(pt)}{n-p+\alpha_q}}}, \quad (28)$$

$$\alpha_q := \left[ \frac{3q}{1-q} \right] + 2, \quad (29)$$

and  $[m]$  denotes the integral part of the number  $m$ . By virtue of (26) and the obvious inequality

$$pq^p |\cos pt - q^p| Z_{q^p}^2(pt) \leq \frac{pq^p}{1-q^p} \leq \frac{q}{1-q}, \quad q \in (0, 1), \quad p \in \mathbb{N}, \quad t \in \mathbb{R}, \quad (30)$$

the quantity under the radical sign in (28) is always positive. Set

$$\begin{aligned} R_{n,p,q,\beta}(\varphi) &:= \mathcal{J}_{n,p,q,\beta}(\varphi) - \\ &- \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^4(t)}{Z_{q^p}(pt) Z_{q,n,p}^2(t)} \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{q^p}(pt) + \frac{\beta\pi}{2} \right) dt = \\ &= \int_0^{2\pi} \Delta(\varphi, t) \frac{Z_q^2(t)}{Z_{q^p}(pt)} \left( 1 - \frac{Z_q^2(t)}{Z_{q,n,p}^2(t)} \right) \cos \left( (n-p+1)t + 2\theta_q(t) - \theta_{q^p}(pt) + \frac{\beta\pi}{2} \right) dt. \end{aligned} \quad (31)$$

To prove equality (27) it suffices to show that the estimate

$$R_{n,p,q,\beta}(\varphi) = O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)} \quad (32)$$

holds, where the quantity  $\delta(p)$  is defined by (11). Indeed, considering the inequality  $|\Delta(\varphi, t)| \leq \omega(|t|)$ , from (31) we get

$$|R_{n,p,q,\beta}(\varphi)| \leq \omega(2\pi) \int_0^{2\pi} \frac{Z_q^2(t)}{Z_{q^p}(pt)} \left| 1 - \frac{Z_q^2(t)}{Z_{q,n,p}^2(t)} \right| dt. \quad (33)$$

After performing elementary transformations and taking into account (30), we find

$$\begin{aligned} \left| 1 - \frac{Z_q^2(t)}{Z_{q,n,p}^2(t)} \right| &= \left| \frac{2q(\cos t - q)Z_q^2(t) - pq^p(\cos pt - q^p)Z_{q^p}^2(pt) + 1 - \alpha_q}{n-p+\alpha_q} \right| < \\ &< \frac{1}{n-p+\alpha_q} \left( 2q |\cos t - q| Z_q^2(t) + pq^p |\cos pt - q^p| Z_{q^p}^2(pt) + \frac{3q}{1-q} + 1 \right) \leq \\ &\leq \frac{1}{n-p+\alpha_q} \left( \frac{2q}{1-q} + \frac{q}{1-q} + \frac{3q}{1-q} + 1 \right) = \frac{O(1)}{(1-q)(n-p+1)}. \end{aligned} \quad (34)$$

From the estimate

$$\frac{Z_q^2(t)}{Z_{q^p}(pt)} = \frac{1}{\sqrt{1 - 2q \cos t + q^2}} \leq \frac{1}{1 - q}, \quad t \in \mathbb{R}, \quad p = 1,$$

and (14) it follows that

$$\frac{Z_q^2(t)}{Z_{q^p}(pt)} = \frac{\sqrt{1 - 2q^p \cos pt + q^{2p}}}{1 - 2q \cos t + q^2} = \frac{O(1)}{(1 - q)^{\delta(p)-1}}, \quad t \in \mathbb{R}, \quad p \in \mathbb{N}. \quad (35)$$

Comparing (33)–(35), we obtain (32), and with it estimate (27).

Consider the function

$$y_1(t) := t + \frac{1}{n - p + \alpha_q} \left( 2\theta_q(t) - \theta_{q^p}(pt) + (1 - \alpha_q)t + \frac{\beta\pi}{2} \right). \quad (36)$$

On the strength of the fact that  $(\theta_{q^p}(pt))' = pq^p(\cos pt - q^p)Z_{q^p}^2(pt)$ , the equality

$$\begin{aligned} y_1'(t) &= 1 + \frac{1}{n - p + \alpha_q} \left( 2q(\cos t - q)Z_q^2(t) - pq^p(\cos pt - q^p)Z_{q^p}^2(pt) + 1 - \alpha_q \right) = \\ &= \frac{Z_q^2(t)}{Z_{q,n,p}^2(t)}, \end{aligned} \quad (37)$$

holds, where  $Z_{q,n,p}(t)$  is defined by (28). In view of (30)

$$-\frac{5q + 1}{1 - q} \leq 2q(\cos t - q)Z_q^2(t) - pq^p(\cos pt - q^p)Z_{q^p}^2(pt) + 1 - \alpha_q < 0. \quad (38)$$

Thus, we see from (26) and (37) that the next two-sided estimate

$$\frac{1}{3} < y_1'(t) < 1 \quad (39)$$

holds. So  $y_1$  has the inverse function  $y(t) = y_1^{-1}(t)$ , whose derivative  $y'$  by (37) satisfies the relation

$$y'(t) = \frac{1}{y_1'(y(t))} = \frac{Z_{q,n,p}^2(y(t))}{Z_q^2(y(t))}. \quad (40)$$

Making the change of variables  $t = y(\tau)$  in (27), we obtain by (37) the relation

$$\begin{aligned} &\mathcal{J}_{n,p,q,\beta}(\varphi) = \\ &= \int_{y_1(0)}^{y_1(2\pi)} \Delta(\varphi, y(\tau)) \frac{Z_q^4(y(\tau))}{Z_{q^p}(py(\tau))Z_{q,n,p}^2(y(\tau))} \frac{Z_{q,n,p}^2(y(\tau))}{Z_q^2(y(\tau))} \cos((n - p + \alpha_q)\tau) d\tau + \\ &\quad + O(1) \frac{\omega(\pi)}{(1 - q)^{\delta(p)}(n - p + 1)} = \\ &= \int_{y_1(0)}^{y_1(2\pi)} \Delta(\varphi, y(\tau)) \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} \cos((n - p + \alpha_q)\tau) d\tau + \\ &\quad + O(1) \frac{\omega(\pi)}{(1 - q)^{\delta(p)}(n - p + 1)}. \end{aligned} \quad (41)$$

We set

$$x_k := \frac{k\pi}{n-p+\alpha_q}, \quad \tau_k := x_k + \frac{\pi}{2(n-p+\alpha_q)}, \quad k \in \mathbb{N} \quad (42)$$

and

$$l_n(\tau) = \begin{cases} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))}, & \tau \in [x_k, x_{k+1}], \quad k = 2, 3, \dots, k_0 - 2, k_0 - 1, \\ 0, & \tau \in [y_1(0), x_2) \cup (x_{k_0}, y_1(2\pi)], \end{cases}$$

where  $k_0$  is an index such that  $\tau_{k_0}$  is the nearest to the left of  $y_1(2\pi)$  root of the function  $\cos((n-p+\alpha_q)\tau)$ . Using this notations, from (41) we get

$$\begin{aligned} \mathcal{J}_{n,p,q,\beta}(\varphi) &= \int_{x_2}^{x_{k_0}} \Delta(\varphi, y(\tau)) l_n(\tau) \cos((n-p+\alpha_q)\tau) d\tau + R_{n,p,q}^{(1)}(\varphi) + R_{n,p,q}^{(2)}(\varphi) + \\ &\quad + O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)}, \end{aligned} \quad (43)$$

where

$$\begin{aligned} R_{n,p,q}^{(1)}(\varphi) &:= \\ &= \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \left( \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \right) \cos((n-p+\alpha_q)\tau) d\tau, \\ R_{n,p,q}^{(2)}(\varphi) &:= \left( \int_{y_1(0)}^{x_2} + \int_{x_{k_0}}^{y_1(2\pi)} \right) \Delta(\varphi, y(\tau)) \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} \cos((n-p+\alpha_q)\tau) d\tau. \end{aligned}$$

Since  $y(t)$  is an increasing function (see (39), (40)), then

$$y(x_{k+1}) \leq y(x_{k_0}) < y(y_1(2\pi)) = 2\pi, \quad k = \overline{2, k_0-1}$$

and so for  $R_{n,p,q}^{(1)}(\varphi)$  the following trivial estimate

$$\begin{aligned} |R_{n,p,q}^{(1)}(\varphi)| &\leq \sum_{k=2}^{k_0-1} \omega(y(x_{k+1})) \int_{x_k}^{x_{k+1}} \left| \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \right| d\tau < \\ &< \omega(2\pi) \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left| \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \right| d\tau \end{aligned} \quad (44)$$

holds.

We will show that

$$\sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left| \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \right| d\tau = \frac{O(1)q}{(1-q)^{\delta(p)}(n-p+1)}. \quad (45)$$

For this purpose, we consider the derivative

$$\left( \frac{Z_q^2(t)}{Z_{q^p}(pt)} \right)' = -2q \sin t \frac{Z_q^4(t)}{Z_{q^p}(pt)} + q^p p \sin pt \frac{Z_q^2(t)}{Z_{q^p}(pt)} =: J_{q,p}^{(1)}(t) + J_{q,p}^{(2)}(t) \quad (46)$$

and estimate separately the summands of the right-hand side of (46). Taking into account (35) and the estimate

$$q|\sin t|Z_q^2(t) = \frac{q|\sin t|}{1 - 2q\cos t + q^2} < \sum_{k=1}^{\infty} q^k = \frac{q}{1-q}, \quad (47)$$

we obtain for  $J_{q,p}^{(1)}(t)$

$$|J_{q,p}^{(1)}(t)| \leq 2q|\sin t|Z_q^2(t) \frac{Z_q^2(t)}{Z_{q^p}(pt)} = O(1) \frac{q}{(1-q)^{\delta(p)}}. \quad (48)$$

If  $p = 1$ , then in view of (47) we have for  $J_{q,p}^{(2)}(t)$

$$|J_{q,1}^{(2)}(t)| < \frac{q}{(1-q)^2}.$$

For any  $p = 2, 3, \dots$ , one easily shows that

$$|J_{q,p}^{(2)}(t)| < \frac{q^p p}{(1-q)^2(1-q^p)} < \frac{q}{(1-q)^3}.$$

Therefore we finally obtain

$$|J_{q,p}^{(2)}(t)| < \frac{q}{(1-q)^{\delta(p)}}, \quad p \in \mathbb{N}. \quad (49)$$

Combining (46), (48) and (49), we arrive at the estimate

$$\left| \left( \frac{Z_q^2(t)}{Z_{q^p}(pt)} \right)' \right| = O(1) \frac{q}{(1-q)^{\delta(p)}}, \quad t \in [0, 2\pi].$$

Since by (39) and (40)  $|y'(t)| < 3$ , applying Lagrange's theorem on finite increments, we find

$$\begin{aligned} & \left| \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \right| = \\ &= O(1) \frac{q}{(1-q)^{\delta(p)}} |y(\tau) - y(\tau_k)| = O(1) \frac{q}{(1-q)^{\delta(p)}} |\tau - \tau_k| = \\ &= O(1) \frac{q}{(1-q)^{\delta(p)}(n-p+\alpha_q)} = \\ &= O(1) \frac{q}{(1-q)^{\delta(p)}(n-p+1)}, \quad \tau \in [x_k, x_{k+1}], \quad k = \overline{2, k_0-1}. \end{aligned} \quad (50)$$

It results from (50) that

$$\sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left| \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} - \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \right| d\tau = O(1) \frac{qx_{k_0}}{(1-q)^{\delta(p)}(n-p+1)}. \quad (51)$$

But since

$$x_{k_0} < y_1(2\pi) = \int_0^{2\pi} y_1'(t) dt + y_1(0) =$$

$$= \int_0^{2\pi} y_1'(t) dt + \frac{\beta\pi}{2(n-p+\alpha_q)} < 2\pi + \frac{\beta\pi}{2} < 4\pi,$$

(45) follows from (51). Estimates (44) and (45) imply

$$|R_{n,p,q}^{(1)}(\varphi)| = O(1) \frac{\omega(\pi)q}{(1-q)^{\delta(p)}(n-p+1)}. \quad (52)$$

Further, considering that

$$x_2 - y_1(0) \leq \frac{2\pi}{n-p+\alpha_q} < \frac{2\pi}{n-p+1}, \quad (53)$$

$$y_1(2\pi) - x_{k_0} \leq \tau_{k_0+1} - x_{k_0} = \frac{3\pi}{2(n-p+\alpha_q)} < \frac{3\pi}{2(n-p+1)} \quad (54)$$

and using (35), we find

$$|R_{n,p,q}^{(2)}(\varphi)| = O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)-1}(n-p+1)}. \quad (55)$$

From (43), (52) and (55) we obtain

$$\begin{aligned} \mathcal{J}_{n,p,q,\beta}(\varphi) &= \int_{x_2}^{x_{k_0}} \Delta(\varphi, y(\tau)) l_n(\tau) \cos((n-p+\alpha_q)\tau) d\tau + \\ &+ O(1) \frac{\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)}, \quad \varphi \in H_\omega, \quad n-p \rightarrow \infty, \end{aligned} \quad (56)$$

where  $O(1)$  is quantity uniformly bounded relative to all parameters under consideration.

Comparing (24), (25) and (56) we conclude that

$$\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) = \frac{q^{n-p+1}}{\pi p} \left( \sup_{\varphi \in H_\omega} |I_{n,p,q,\beta}(\varphi)| + \frac{O(1)\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)} \right), \quad (57)$$

in which

$$\begin{aligned} I_{n,p,q,\beta}(\varphi) &:= \int_{x_2}^{x_{k_0}} \Delta(\varphi, y(\tau)) l_n(\tau) \cos((n-p+\alpha_q)\tau) d\tau = \\ &= \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos((n-p+\alpha_q)\tau) d\tau. \end{aligned} \quad (58)$$

*Step 2.* Using formula (57) we find an upper bound for  $\mathcal{E}(C_\beta^q H_\omega; V_{n,p})$ .

With this goal, dividing each integral

$$\int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos((n-p+\alpha_q)\tau) d\tau, \quad k = \overline{2, k_0-1}$$

into two integrals over  $(x_k, \tau_k)$  and  $(\tau_k, x_{k+1})$ , and setting  $z = 2\tau_k - \tau$  in the last integral, we obtain

$$\left| \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos((n-p+\alpha_q)\tau) d\tau \right| =$$

$$= \left| \int_{x_k}^{\tau_k} (\varphi(y(\tau)) - \varphi(y(2\tau_k - \tau))) \cos((n - p + \alpha_q)\tau) d\tau \right|, \quad k = \overline{2, k_0 - 1}. \quad (59)$$

We choose  $c_k \in [x_k, x_{k+1}]$  such that

$$y'(c_k) = \max_{\tau \in [x_k, x_{k+1}]} y'(\tau).$$

Then for any  $\tau \in [x_k, \tau_k]$

$$y(2\tau_k - \tau) - y(\tau) = \int_{\tau}^{2\tau_k - \tau} y'(x) dx \leq y'(c_k)(2\tau_k - 2\tau)$$

and consequently

$$\begin{aligned} & |\varphi(y(\tau)) - \varphi(y(2\tau_k - \tau))| \leq \\ & \leq \omega(y(2\tau_k - \tau) - y(\tau)) \leq \omega(2y'(c_k)(\tau_k - \tau)), \quad k = \overline{2, k_0 - 1}. \end{aligned} \quad (60)$$

From (59), in view of (60), we find that

$$\begin{aligned} & \left| \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos((n - p + \alpha_q)\tau) d\tau \right| \leq \\ & \leq \int_{x_k}^{\tau_k} \omega(2y'(c_k)(\tau_k - \tau)) |\cos((n - p + \alpha_q)\tau)| d\tau \leq \\ & \leq \frac{1}{n - p + \alpha_q} \int_0^{\pi/2} \omega\left(\frac{2y'(c_k)t}{n - p + \alpha_q}\right) \sin t dt < \\ & < \frac{1}{n - p + \alpha_q} \int_0^{\pi/2} \omega\left(\frac{2y'(c_k)t}{n - p + 1}\right) \sin t dt = \\ & = \frac{1}{n - p + \alpha_q} \int_0^{\pi/2} \omega\left(\frac{2t}{n - p + 1}\right) \sin t dt + \\ & + \frac{O(1)}{n - p + \alpha_q} \max_{t \in (0, \pi/2]} \left| \omega\left(\frac{2y'(c_k)t}{n - p + 1}\right) - \omega\left(\frac{2t}{n - p + 1}\right) \right|, \quad k = \overline{2, k_0 - 1}. \end{aligned} \quad (61)$$

Because for any convex upwards modulus of continuity  $\omega$

$$\omega(b) - \omega(a) \leq \omega(a) \frac{b - a}{a}, \quad 0 < a < b, \quad (62)$$

then taking into consideration that by (39) and (40)  $y'(c_k) > 1$ , we have

$$\begin{aligned} & \max_{t \in (0, \pi/2]} \left| \omega\left(\frac{2y'(c_k)t}{n - p + 1}\right) - \omega\left(\frac{2t}{n - p + 1}\right) \right| \leq \\ & \leq \omega\left(\frac{\pi}{n - p + 1}\right) (y'(c_k) - 1), \quad k = \overline{2, k_0 - 1}. \end{aligned} \quad (63)$$

By virtue of (40),

$$y'(c_k) - 1 = \frac{Z_{q,n,p}^2(y(c_k))}{Z_q^2(y(c_k))} - 1 = \frac{\alpha_q - 1 - \lambda_{p,q}(c_k)}{n - p + 1 + \lambda_{p,q}(c_k)}, \quad (64)$$

where

$$\lambda_{p,q}(c_k) = 2q(\cos y(c_k) - q)Z_q^2(y(c_k)) - pq^p(\cos py(c_k) - q^p)Z_{q^p}^2(py(c_k)).$$

In view of (30),

$$|\lambda_{p,q}(c_k)| \leq \frac{3q}{1-q}.$$

Hence it follows from (64), taking into account (26), that

$$y'(c_k) - 1 \leq \frac{\alpha_q - 1 + \frac{3q}{1-q}}{n-p+1 - \frac{3q}{1-q}} < \frac{6}{(1-q)(n-p+1 - \frac{3q}{1-q})} < \frac{12}{(1-q)(n-p+1)}. \quad (65)$$

Comparing (61), (63) and (65), we obtain

$$\begin{aligned} & \left| \int_{x_k}^{x_{k+1}} \Delta(\varphi, y(\tau)) \cos(n-p+\alpha_q)\tau \, d\tau \right| \leq \\ & \leq \frac{1}{n-p+\alpha_q} \left( \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t \, dt + \right. \\ & \left. + \frac{O(1)}{(1-q)(n-p+1)} \omega\left(\frac{1}{n-p+1}\right) \right), \quad k = \overline{2, k_0-1}. \end{aligned} \quad (66)$$

Applying to each integral in (58) estimate (66), we have

$$\begin{aligned} |I_{n,p,q,\beta}(\varphi)| & \leq \frac{1}{n-p+\alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t \, dt + \\ & + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} \omega\left(\frac{1}{n-p+1}\right). \end{aligned} \quad (67)$$

Let us show that

$$\frac{\pi}{n-p+\alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} = \int_0^{2\pi} \frac{Z_q^2(t)}{Z_{q^p}(pt)} \, dt + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)}. \quad (68)$$

For this, we represent the left-hand side of (68) as

$$\frac{\pi}{n-p+\alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} = \int_{y_1(0)}^{y_1(2\pi)} \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} \, d\tau + R_{n,p,q}^{(1)} + R_{n,p,q}^{(2)}, \quad (69)$$

where

$$\begin{aligned} R_{n,p,q}^{(1)} & := - \left( \int_{y_1(0)}^{x_2} + \int_{x_{k_0}}^{y_1(2\pi)} \right) \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} \, d\tau, \\ R_{n,p,q}^{(2)} & := \sum_{k=2}^{k_0-1} \int_{x_k}^{x_{k+1}} \left( \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} - \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} \right) \, d\tau. \end{aligned}$$

By virtue of (35), (53) and (54)

$$R_{n,p,q}^{(1)} = \frac{O(1)}{(1-q)^{\delta(p)-1}(n-p+1)}, \quad (70)$$

and by (45)

$$R_{n,p,q}^{(2)} = \frac{O(1)q}{(1-q)^{\delta(p)}(n-p+1)}. \quad (71)$$

Combining (69)–(71), we can write

$$\begin{aligned} \frac{\pi}{n-p+\alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} &= \int_{y_1(0)}^{y_1(2\pi)} \frac{Z_q^2(y(\tau))}{Z_{q^p}(py(\tau))} d\tau + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} = \\ &= \int_0^{2\pi} \frac{Z_q^2(t)}{Z_{q^p}(pt)} dt + \int_0^{2\pi} \frac{Z_q^2(t)}{Z_{q^p}(pt)} (y_1'(t) - 1) dt + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)}. \end{aligned} \quad (72)$$

But in view of (37) and (38)

$$|y_1'(t) - 1| < \frac{6}{(1-q)(n-p+1)}.$$

Thus, in consideration of (35), we obtain from (72) equality (68).

Estimates (57), (67) and (68) imply that as  $n-p \rightarrow \infty$

$$\begin{aligned} \mathcal{E}(C_\beta^q H_\omega; V_{n,p}) &\leq \frac{q^{n-p+1}}{\pi^2 p} K_{p,q} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t dt + \\ &+ O(1) \frac{q^{n-p+1} \omega(\pi)}{(1-q)^{\delta(p)} p (n-p+1)}, \end{aligned} \quad (73)$$

where

$$K_{p,q} = \int_0^{2\pi} \frac{Z_q^2(t)}{Z_{q^p}(pt)} dt, \quad (74)$$

and  $O(1)$  is a quantity uniformly bounded in  $n, p, q, \omega$  and  $\beta$ .

*Step 3.* We now show that (73) is an equality. For this, in view of (57), it is sufficient to show that there exists a function  $\varphi^* \in H_\omega$  such that

$$I_{n,p,q,\beta}(\varphi^*) = \frac{K_{p,q}}{\pi} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t dt + \frac{O(1)\omega(\pi)}{(1-q)^{\delta(p)}(n-p+1)}, \quad (75)$$

where  $I_{n,p,q,\beta}(\varphi^*)$  is defined by (58). To this end, we set

$$\begin{aligned} \varphi_i(t) &:= \begin{cases} \frac{1}{2}\omega(2y_1(t) - 2\tau_i), & t \in [y(\tau_i), y(x_{i+1})], \\ \frac{1}{2}\omega(2\tau_{i+1} - 2y_1(t)), & t \in [y(x_{i+1}), y(\tau_{i+1})], \end{cases} \quad i = \overline{s, k_0 - 1}, \\ s &= \begin{cases} 2, & \text{if } k_0 \text{ is odd,} \\ 3, & \text{if } k_0 \text{ is even,} \end{cases} \\ \tilde{\varphi}(t) &:= (-1)^{i+1} \varphi_i(t), \quad t \in [y(\tau_i), y(\tau_{i+1})], \quad i = \overline{s, k_0 - 1}. \end{aligned}$$

Since  $\tau_{k_0} \leq y_1(2\pi)$  and by (42), (36) and  $\beta \in [0, 4)$ , the inequality  $\tau_s > y_1(0)$  holds, it follows that  $y(\tau_{k_0}) \leq 2\pi$  and  $y(\tau_s) > 0$ . Consider the function

$$\varphi^*(t) := \begin{cases} \tilde{\varphi}(t), & t \in [y(\tau_s), y(\tau_{k_0})], \\ 0, & t \in [0, 2\pi] \setminus [y(\tau_s), y(\tau_{k_0})], \end{cases} \quad \varphi^*(t) = \varphi^*(t + 2\pi). \quad (76)$$

We show that, if (26) holds, then  $\varphi^* \in H_\omega$ . The construction of  $\varphi^*$  shows that it suffices to establish the inequality

$$|\varphi^*(t') - \varphi^*(t'')| \leq \omega(t'' - t'),$$

where  $t' \in [y(x_i), y(\tau_i)]$  and  $t'' \in [y(\tau_i), y(x_{i+1})]$ ,  $i = \overline{s+1, k_0-1}$ .

In view of the convexity (upwards) of the modulus of continuity

$$\frac{1}{2}(\omega(t_1) + \omega(t_2)) \leq \omega\left(\frac{t_1 + t_2}{2}\right).$$

Then, considering that by (39)  $y'_1 \in (\frac{1}{3}, 1)$ , we get

$$\begin{aligned} |\varphi^*(t') - \varphi^*(t'')| &= |\tilde{\varphi}(t') - \tilde{\varphi}(t'')| = \varphi_{i-1}(t') + \varphi_i(t'') = \\ &= \frac{1}{2}(\omega(2\tau_i - 2y_1(t')) + \omega(2y_1(t'') - 2\tau_i)) \leq \\ &\leq \omega(y_1(t'') - y_1(t')) \leq \omega(t'' - t'), \quad i = \overline{s+1, k_0-1}. \end{aligned}$$

Let us now verify that  $\varphi^*(t)$  is the desired function, which means that  $\varphi^*(t)$  satisfies (75). Since by (76)

$$\varphi^*(y(\tau)) = \begin{cases} \frac{(-1)^{i+1}}{2}\omega(2\tau - 2\tau_i), & \tau \in [\tau_i, x_{i+1}], \\ \frac{(-1)^{i+1}}{2}\omega(2\tau_{i+1} - 2\tau), & \tau \in [x_{i+1}, \tau_{i+1}], \end{cases} \quad i = \overline{s, k_0-1},$$

it follows that

$$\begin{aligned} &\int_{x_k}^{x_{k+1}} \Delta(\varphi^*, y(\tau)) \cos((n - p + \alpha_q)\tau) d\tau = \\ &= \frac{(-1)^k}{2} \left( \int_{x_k}^{\tau_k} \omega(2\tau_k - 2\tau) \cos((n - p + \alpha_q)\tau) d\tau - \right. \\ &\quad \left. - \int_{\tau_k}^{x_{k+1}} \omega(2\tau - 2\tau_k) \cos((n - p + \alpha_q)\tau) d\tau \right) = \\ &= \int_0^{\pi/2(n-p+\alpha_q)} \omega(2t) \sin((n - p + \alpha_q)\tau) d\tau = \\ &= \frac{1}{n - p + \alpha_q} \int_0^{\pi/2} \omega\left(\frac{2t}{n - p + \alpha_q}\right) \sin t dt, \quad k = \overline{s+1, k_0-1}. \end{aligned} \quad (77)$$

By (77) and (35), we obtain

$$I_{n,p,q,\beta}(\varphi^*) = \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \int_{x_k}^{x_{k+1}} \Delta(\varphi^*, y(\tau)) \cos((n - p + \alpha_q)\tau) d\tau =$$

$$\begin{aligned}
&= \frac{1}{n-p+\alpha_q} \sum_{k=2}^{k_0-1} \frac{Z_q^2(y(\tau_k))}{Z_{q^p}(py(\tau_k))} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+\alpha_q}\right) \sin t \, dt + \\
&\quad + \frac{O(1)}{(n-p+1)(1-q)^{\delta(p)-1}} \omega\left(\frac{1}{n-p+1}\right). \tag{78}
\end{aligned}$$

From (78), in view of (68) and (74), we find

$$\begin{aligned}
I_{n,p,q,\beta}(\varphi^*) &= \frac{K_{p,q}}{\pi} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+\alpha_q}\right) \sin t \, dt + \\
&\quad + \frac{O(1)}{(n-p+1)(1-q)^{\delta(p)}} \omega\left(\frac{1}{n-p+1}\right). \tag{79}
\end{aligned}$$

Based on (62) and (29), we can write

$$\begin{aligned}
&\max_{t \in (0, \pi/2]} \left| \omega\left(\frac{2t}{n-p+\alpha_q}\right) - \omega\left(\frac{2t}{n-p+1}\right) \right| = \\
&= O(1) \frac{\alpha_q - 1}{n-p+1} \omega\left(\frac{1}{n-p+1}\right) = \frac{O(1)}{(n-p+1)(1-q)} \omega\left(\frac{1}{n-p+1}\right). \tag{80}
\end{aligned}$$

Comparing (79), (80) and taking into account that by (35),

$$K_{p,q} = \frac{O(1)}{(1-q)^{\delta(p)-1}}, \tag{81}$$

we arrive at (75).

Combining (57) and (75), we obtain the estimate

$$\begin{aligned}
\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) &\geq \frac{q^{n-p+1}}{\pi^2 p} K_{p,q} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t \, dt + \\
&\quad + O(1) \frac{q^{n-p+1} \omega(\pi)}{(1-q)^{\delta(p)} p (n-p+1)}, \tag{82}
\end{aligned}$$

in which  $O(1)$  is a quantity uniformly bounded in  $n, p, q, \omega$  and  $\beta$ . From (73) and (82) we get asymptotic formula (9). Theorem 1 is proved.  $\blacksquare$

*Proof of Theorem 3.* Since the sequence  $e_k(\omega)$  is monotonically decreasing (see (7)), from (2) and (6) it follows that

$$\begin{aligned}
\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) &\leq \frac{1}{p} \sum_{k=n-p+1}^n \mathcal{E}(C_\beta^q H_\omega; S_{k-1}) \leq \\
&\leq \frac{q^{n-p+1}}{p} \left( \frac{4}{\pi^2} \frac{1-q^p}{1-q} \mathbf{K}(q) e_{n-p+1}(\omega) + \frac{O(1)}{(1-q)^{\delta(p)} (n-p+1)} \omega\left(\frac{1}{n-p+1}\right) \right). \tag{83}
\end{aligned}$$

On the other hand, if we analyze the proof of (57), it is easy to see that for any function  $\varphi \in H_\omega$  the estimate

$$\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) \geq \frac{q^{n-p+1}}{\pi p} \left( |I_{n,p,q,\beta}(\varphi)| + \frac{O(1)}{(1-q)^{\delta(p)} (n-p+1)} \|\Delta(\varphi, \cdot)\|_C \right) \tag{84}$$

holds. For the function  $\varphi^*(t)$  defined by (76), we have from (79)–(81) that

$$\begin{aligned} I_{n,p,q,\beta}(\varphi^*) &= \frac{K_{p,q}}{\pi} \int_0^{\pi/2} \omega\left(\frac{2t}{n-p+1}\right) \sin t \, dt + \\ &+ \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} \omega\left(\frac{1}{n-p+1}\right). \end{aligned} \quad (85)$$

Since  $\|\Delta(\varphi^*, \cdot)\|_C = \frac{1}{2}\omega(\frac{\pi}{n-p+\alpha_q})$ , comparing (84) and (85), we obtain

$$\mathcal{E}(C_\beta^q H_\omega; V_{n,p}) \geq \frac{q^{n-p+1}}{p} \left( \frac{K_{p,q}}{\pi^2} e_{n-p+1}(\omega) + \frac{O(1)}{(1-q)^{\delta(p)}(n-p+1)} \omega\left(\frac{1}{n-p+1}\right) \right). \quad (86)$$

From (83), (86) and (15), relation (16) follows. Theorem 3 is proved.  $\blacksquare$

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